Matrix Analysis of a Nonuniform Beam Column on Multi-Supports

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IN a previous paper,[1] the author presented a "transfer matrix"[2] approach for a beam column under various end conditions. This paper extends the method of beam columns to multi-support systems. For an element of a beam column between x_i and x_{i+1}, the relation between the output and the input is expressed by the transfer matrix equation

$$|y|_{i+1} = |A_{i+1}|\,|y|_i \tag{1}$$

where $|y|$ represents the column matrix $\{SM\varphi y\}$, and S_i, M_i, φ_i, y_i are the shear, moment, slope, and deflection at x_i.

The transfer matrix is

$$A_{i+1} = \begin{vmatrix} \cos\bar{\alpha}_{i+1}l_{i+1} & -\bar{\alpha}_{i+1}\sin\bar{\alpha}_{i+1}l_{i+1} & -\bar{\alpha}_{i+1}^2(EI)_{i+1}\cos\bar{\alpha}_{i+1}l_{i+1} & 0 \\ \dfrac{\sin\bar{\alpha}_{i+1}l_{i+1}}{\bar{\alpha}_{i+1}} & \cos\bar{\alpha}_{i+1}l_{i+1} & -\bar{\alpha}(EI)_{i+1}\sin\bar{\alpha}_{i+1}l_{i+1} & 0 \\ \dfrac{1-\cos\bar{\alpha}_{i+1}l_{i+1}}{\bar{\alpha}_{i+1}^2(EI)_{i+1}} & \dfrac{\sin\bar{\alpha}_{i+1}l_{i+1}}{\bar{\alpha}_{i+1}(EI)_{i+1}} & \cos\bar{\alpha}_{i+1}l_{i+1} & 0 \\ \dfrac{\bar{\alpha}_{i+1}l_{i+1}-\sin\bar{\alpha}_{i+1}l_{i+1}}{\bar{\alpha}_{i+1}^3(EI)_{i+1}} & \dfrac{1-\cos\bar{\alpha}_{i+1}l_{i+1}}{\bar{\alpha}_{i+1}^2(EI)_{i+1}} & \dfrac{-\sin\bar{\alpha}_{i+1}l_{i+1}}{\bar{\alpha}_{i+1}} & 1 \end{vmatrix} \tag{2}$$

Consider a beam column with multiple supports (Fig. 1). The relation between stations 0 and $1L$ is

$$\begin{vmatrix} S \\ M \\ \varphi \\ y \end{vmatrix}_{1L} = \begin{vmatrix} \cos\alpha_1 l_1 & -\alpha_1\sin\alpha_1 l_1 & -\alpha_1^2(EI)_1\cos\alpha_1 l_1 & 0 \\ \dfrac{\sin\alpha_1 l_1}{\alpha_1} & \cos\alpha_1 l_1 & -\alpha_1(EI)_1\sin\alpha_1 l_1 & 0 \\ \dfrac{1-\cos\alpha_1 l_1}{\alpha_1^2(EI)_1} & \dfrac{\sin\alpha_1 l_1}{\alpha_1(EI)_1} & \cos\alpha_1 l_1 & 0 \\ \dfrac{\alpha_1 l_1 - \sin\alpha_1 l_1}{\alpha_1^3(EI)_1} & \dfrac{1-\cos\alpha_1 l_1}{\alpha_1^2(EI)_1} & \dfrac{-\sin\alpha_1 l_1}{\alpha_1} & 1 \end{vmatrix} \begin{vmatrix} S \\ M \\ \varphi \\ y \end{vmatrix}_{0} \tag{3}$$

For a free end, S and M are both zero; therefore, Eq. (3) reduces to

$$\begin{vmatrix} S \\ M \\ \varphi \\ y \end{vmatrix}_{1L} = \begin{vmatrix} -\alpha_1^2(EI)_1\cos\alpha_1 l_1 & 0 \\ -\alpha_1(EI)_1\sin\alpha_1 l_1 & 0 \\ \cos\alpha_1 l_1 & 0 \\ \dfrac{-\sin\alpha_1 l_1}{\alpha_1} & 1 \end{vmatrix} \begin{vmatrix} \varphi \\ y \end{vmatrix}_{0} \tag{4}$$

From Marguerre,[3] the relation between the right and left sides of the spring support is

$$\begin{vmatrix} S \\ M \\ \varphi \\ y \end{vmatrix}_{1R} = \begin{vmatrix} 1 & 0 & 0 & k \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} S \\ M \\ \varphi \\ y \end{vmatrix}_{1L} \tag{5}$$

Thus, combining Eqs. (4) and (5) by matrix multiplication,

$$\begin{vmatrix} S \\ M \\ \varphi \\ y \end{vmatrix}_{1R} = \begin{vmatrix} -\alpha_1^2(EI)_1\cos\alpha_1 l_1 - \left(\dfrac{1}{\alpha_1}\right)k\sin\alpha_1 l_1 & k \\ -\alpha_1(EI)_1\sin\alpha_1 l_1 & 0 \\ \cos\alpha_1 l_1 & 0 \\ \dfrac{-\sin\alpha_1 l_1}{\alpha_1} & 1 \end{vmatrix} \begin{vmatrix} \varphi \\ y \end{vmatrix}_{0} \tag{6}$$

Similarly, the relation between station $2L$ and $1R$ is

$$\begin{vmatrix} S \\ M \\ \varphi \\ y \end{vmatrix}_{2L} = \begin{vmatrix} \cos\alpha_2 l_2 & -\alpha_2\sin\alpha_2 l_2 & -\alpha_2^2(EI)_2\cos\alpha_2 l_2 & 0 \\ \dfrac{\sin\alpha_2 l_2}{\alpha_2} & \cos\alpha_2 l_2 & -\alpha_2(EI)_2\sin\alpha_2 l_2 & 0 \\ \dfrac{1-\cos\alpha_2 l_2}{\alpha_2^2(EI)_2} & \dfrac{\sin\alpha_2 l_2}{\alpha_2(EI)_2} & \cos\alpha_2 l_2 & 0 \\ \dfrac{\alpha_2 l_2 - \sin\alpha_2 l_2}{\alpha_2^3(EI)_2} & \dfrac{1-\cos\alpha_2 l_2}{\alpha_2^2(EI)_2} & \dfrac{-\sin\alpha_2 l_2}{\alpha_2} & 1 \end{vmatrix} \begin{vmatrix} S \\ M \\ \varphi \\ y \end{vmatrix}_{1R} \tag{7}$$

Matrix-multiplying Eq. (6) by Eq. (7), one has

$$\begin{vmatrix} S \\ M \\ \varphi \\ y \end{vmatrix}_{2L} = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{vmatrix} \begin{vmatrix} \varphi \\ y \end{vmatrix}_{0} \tag{8}$$

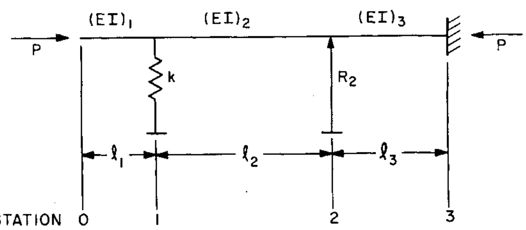

Fig. 1 Beam column on multiple supports

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where

$$b_{11} = -\cos\alpha_2 l_2[\alpha_1^2(EI)_1 \cos\alpha_1 l_1 + (1/\alpha_1)k \sin\alpha_1 l_1] + \alpha_1\alpha_2 \sin\alpha_1 l_1 \sin\alpha_2 l_2(EI)_1 - \cos\alpha_1 l_1[\alpha_2^2(EI)_2 \cos\alpha_2 l_2]$$

$$b_{21} = -(\sin\alpha_2 l_2/\alpha_2)[\alpha_1^2(EI)_1 \cos\alpha_1 l_1] - \cos\alpha_2 l_2 [\alpha_1(EI)_1 \sin\alpha_1 l_1] - \cos\alpha_1 l_1 [\alpha_2(EI)_2 \sin\alpha_2 l_2] - (\sin\alpha_2 l_2/\alpha_2)[k \sin\alpha_1 l_1/\alpha_1]$$

$$\begin{matrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix}$$

$$b_{42} = k[(\alpha_2 l_2 - \sin\alpha_2 l_2)/\alpha_2^3(EI)_2] + 1$$

At station 2, the deflections $y_{2L} = y_{2R}$ are both equal to zero. Expand Eq. (8) for the y_{2L} term and then express φ_0 in terms of y_0.

The matrix equation (8) reduces to

$$\begin{vmatrix} S \\ M \\ \varphi \\ y \end{vmatrix}_{2L} = \begin{vmatrix} b_{12} - b_{11}(b_{42}/b_{41}) \\ b_{22} - b_{21}(b_{42}/b_{44}) \\ b_{32} - b_{31}(b_{42}/b_{41}) \\ 0 \end{vmatrix} \begin{vmatrix} y_0 \end{vmatrix} \quad (9)$$

From Fig. 1, the total shear at station 2 is $R_2 + S_2$. In matrix form, this becomes

$$\begin{vmatrix} S \\ M \\ \varphi \\ y \end{vmatrix}_{2R} = \begin{vmatrix} c_{11} & 1 \\ c_{21} & 0 \\ c_{31} & 0 \\ c_{41} & 0 \end{vmatrix} \begin{vmatrix} y_0 \\ R_2 \end{vmatrix} \quad (10)$$

In a similar manner, the transfer matrix between station $2R$ and 3 is

$$\begin{vmatrix} S \\ M \\ \varphi \\ y \end{vmatrix}_{3} = \begin{vmatrix} \cos\alpha_3 l_3 & -\alpha_3 \sin\alpha_3 l_3 & -\alpha_3^2(EI)_3 \cos\alpha_3 l_3 & 0 \\ \dfrac{\sin\alpha_3 l_3}{\alpha_3} & \cos\alpha_3 l_3 & -\alpha_3(EI)_3 \sin\alpha_3 l_3 & 0 \\ \dfrac{1-\cos\alpha_3 l_3}{\alpha_3^2(EI)_3} & \dfrac{\sin\alpha_3 l_3}{\alpha_3(EI)_3} & \cos\alpha_3 l_3 & 0 \\ \dfrac{\alpha_3 l_3 - \sin\alpha_3 l_3}{\alpha_3^3(EI)_3} & \dfrac{1-\cos\alpha_3 l_3}{\alpha_3^2(EI)_3} & -\dfrac{\sin\alpha_3 l_3}{\alpha_3} & 1 \end{vmatrix} \begin{vmatrix} S \\ M \\ \varphi \\ y \end{vmatrix}_{2R} \quad (11)$$

Multiplying matrix Eq. (10) by matrix Eq. (11), the following is obtained:

$$\begin{vmatrix} S \\ M \\ \varphi \\ y \end{vmatrix}_{3} = \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \\ d_{31} & d_{32} \\ d_{41} & d_{42} \end{vmatrix} \begin{vmatrix} \varphi_0 \\ R_2 \end{vmatrix} \quad (12)$$

where

$$d_{11} = c_{11} \cos\alpha_3 l_3 - c_{21}\alpha_3 \sin\alpha_3 l_3 - c_{31}[\alpha_3^2(EI)_3 \cos\alpha_3 l_3]$$

$$d_{21} = (c_{11} \sin\alpha_3 l_3/\alpha_3) + c_{21} \cos\alpha_3 l_3 - c_{31}[\alpha_3(EI)_3 \sin\alpha_3 l_3]$$

$$\begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix}$$

$$d_{42} = (\alpha_3 l_3 - \sin\alpha_3 l_3)/\alpha_3^3(EI)_3$$

At station 3 for a fixed end $\varphi_3 = y_3 = 0$; thus in matrix form

$$\begin{vmatrix} 0 \\ 0 \end{vmatrix} = \begin{vmatrix} d_{31} & d_{32} \\ d_{41} & d_{42} \end{vmatrix} \begin{vmatrix} y_0 \\ R_2 \end{vmatrix} \quad (13)$$

For other than a trivial solution, the system in Eq. (13) has solutions different from zero if the determinant of the system vanishes. Expanding Eq. (13), the buckling load (P) may be determined when

$$d_{31}d_{42} - d_{41}d_{32} - 0 \quad (14)$$

References

[1] Saunders, H., "Beam column of nonuniform sections by matrix methods," J. Aerospace Sci. **28**, 740–741 (1961).

[2] Pestel, E., "Dynamics of structures by transfer matrices," Final Report, Publ. Techn. Hochschule Hannover (June 1961).

[3] Marguerre, K., "Vibration and stability problems of beams treated by matrices," J. Math. Phys. **35**, 28–43 (April 1956).

Free Vibration of a Damped Semi-Elliptical Plate and a Quarter-Elliptical Plate

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Nomenclature

x, y = rectangular coordinates, in.
a, b = major and minor axes of the elliptical plates, in.
w = deflection of the plate, in.
h = thickness of the plate, in.
D = $Eh^3/12(1 - \nu^2)$ = flexural rigidity, lb-in.
E = Young's modulus of elasticity
ν = Poisson's ratio
ρ = mass density of the material, lb-sec^2-in.$^{-4}$
k = damping coefficient, lb-sec-in.$^{-1}$
t = time
w = natural frequency of the system, rad/sec

Subscripts

t, tt, n = derivatives with respect to n and t

An ordinary product solution and the Galerkin method are used as outlined by Stanisic[1] and McNitt[2] to compute the lowest natural frequency of the normal modes of free vibration of a semi-elliptical and a quarter-elliptical plate, both of which are clamped on their boundaries. The classical small-deflection theory is assumed to be valid, and the influence of rotatory inertia is neglected.

Formulation and Solution of the Problem

Because of the shape of the boundaries of the plates considered, difficulties arise for integral transform techniques. However, in the aerospace and ship industries, plates of various shapes occur. For this reason the following approximate solution is given.

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